CHAPTER 22

COMBINATORICS

KEY CONCEPTS AND THEORY

Permutation and Combination is a wonderful topic to learn. This topic is all about counting. In all other topics, we need to get to **an answer**, be it average, or speed, or angle, or LCM. In this topic, we will mostly concern ourselves with the idea of counting the number of ways of doing something. In most scenarios we will be enumerating possibilities rather than solving to get an answer. Before we go further, let us define a few ground rules.

1. This topic is based on the simple idea of counting. So, from now on, we will avoid the terms ‘permutation’ and ‘combination’.
2. We will build the theory for this topic mostly with examples.
3. Wherever possible, we will list and count. Especially, if the answer is less than 10 we will shoot to list out all possibilities.
4. ${}^nC_r$, ${}^nP_r$, n! will be resorted to only if absolutely necessary. Counting errors happen when we look to force–fit an idea on to a question. We will use ${}^nP_r$, ${}^nC_r$ etc, when we hit the framework. We will not use these as starting points.

Now let us move to first set of questions.

Ram wants to travel from Chennai to Kolkatta (to join IIM Kolkatta). He wants to take only 1 piece of baggage with him. He has 3 types of suitcases and 4 types of back packs. In how many ways can he select his luggage?

This is straightforward. Out of the seven pieces available, he has to select exactly one. If he has suitcases S1, S2, S3 and bags B1, B2, B3 and B4, he can pick one of these 7. So, there are 7 options.

After trying to fit in his luggage, Ram realizes that he needs to carry two pieces of baggage. He plans to carry one suitcase and one backpack. In how many ways can he select his baggage now?

He can select S1 or S2 or S3 and B1, B2, B3 or B4. Totally he has 12 options now.

- S1B1, S1B2, S1B3, S1B4
- S2B1, S2B2, S2B3, S2B4
- S3B1, S3B2, S3B3, S3B4

Fundamental Rule of Counting

If there are ‘m’ ways of doing a task and ‘n’ ways of doing another task, then there are m ×n ways of doing both.

If a process can be broken into multiple steps and you have to do each of the many steps, then the total number of ways of doing the process = product of number of ways of doing each step.

AND => ×

Rule of Sum

If there are m ways of doing a task and n ways of doing another task, then the number of ways of doing either of these two (but not both) = m + n. In other words, if we have to do one thing OR another thing, the number of ways = sum of the number of ways of doing each step

OR => +

Number of ways Ram can choose 1 suitcase = 3

Number of ways Ram can choose 1 bag = 4

Number of ways he can choose 1 suitcase OR 1 bag = 3 + 4 = 7

Number of ways he can choose 1 suitcase AND 1 bag = 3 × 4 = 12

Let us take one more example and build counting ideas.

John, who is in charge of creating number plates for cars, has three colors at his disposal – black, white and yellow. He has to paint the number plate with one color and the letters and numbers in another color. In how many ways can he create a number plate?

Again, let us list and count.

1. Black plate, white letters
2. Black plate, yellow letters
3. White plate, yellow letters
4. White plate, black letters
5. Yellow plate, white letters
6. Yellow plate, black letters

There are totally six ways.

Mike has three types of flowers with him – roses, lilies and violets. Mike decides he has to create a bouquet that has two distinct flowers. In how many ways can he create this?

1. Rose and Lily
2. Lily and Violet
3. Rose and Violet

What is the difference?

In the number plate example, we select 2 colors out of 3, in the bouquet example we select 2 flowers out of 3. There are 6 ways of doing the former, but only 3 ways for the latter. What is the difference?

The Idea of Order

The difference is characterised by this term called 'order'. In the number plate example, 'black plate–white letters' is different from 'white plate–black letters'; whereas in the bouquet example, 'rose–lily is the same as lily–rose'.

If in a type of selection 'AB' is different from 'BA', then order is important. If 'AB' is same as 'BA' then order is not important. Essentially, if the reason for which the selections are made is the same, then order does not matter. In the bouquet example, the flowers are chosen for the 'same' bouquet. But in the number plate example, one color is for the <u>background</u> and the other color is for <u>the letters and numbers</u>. Here the reasons are different.

Mike has three types of flowers with him – roses, lilies and violets. Mike decides he has to create a bouquet that has two flowers. In how many ways can he create this?

RL, LV, RV, RR, LL, VV

What is the difference between selecting two distinct flowers and two flowers?

The Idea of Repetition

This difference is based on the idea of repetition. If the same element can be selected again, then we allow 'repetition'. If we are selecting two flowers, then we can select rose and rose. However, if we are looking for two distinct flowers, then rose–rose is not to be counted.

If in a type of selection 'AA' is permitted and should be counted, then repetition is allowed. If 'AA' cannot be a legitimate selection, repetition is not allowed.

Understanding the idea of Repetition

A teacher has a chocolate, a biscuit and a cold–drink with her. She says that whoever gets a question correct would get one of the three as award.

Ram gets the first question right. How many options does he have of choosing his award? **3**

Krish gets the second question right. How many options does he have of choosing his award? **2**

John gets the third question right. He has only **one** option for choosing the award.

Next day, the teacher follows a new awards scheme. She gives Star rating, Circle rating and Square rating to students who get questions right.

Ram gets the first question right. How many options does he have of choosing his award? **3**

Krish gets the second question right. How many options does he have of choosing his award? **3**

John gets the third question right. How many options does he have of choosing his award? **3**

In the first instance, selection is made without repetition. In the second instance, repetition is allowed. If repetition is not allowed, the choice–set shrinks with every selection that is made. If repetition is allowed, then the choice–set remains the same. At every stage, there are the same n objects to choose from.

Rearrangements

Ram, Krishna and John decide to take part in a race. If there is to be no tie, in how many ways can the three positions on the race be determined?

1	Ram	Ram	John	John	Krishna	Krishna
2	Krishna	John	Ram	Krishna	John	Ram
3	John	Krishna	Krishna	Ram	Ram	John

For the first slot, there are 3 options, for the second one there are two options, for the third there is only one option. Totally, there are $3 \times 2 \times 1 = 6 = 3!$ options.

The number of ways of rearranging r objects is given by r!

ILLUSTRATIVE EXAMPLES

1. A school plans to paint the three floors of its building. The painter can choose from the colour red, blue, yellow, and orange for each floor in the building. In how many ways can he choose the colours to paint the building?

EXPLANATION

The painter can choose from four colors for the first floor.

The painter can choose from four colors for the second floor.

The painter can choose from four colors for the third floor.

Total number of options = $4 \times 4 \times 4 = 64$

2. A school plans to paint the three floors in its building. The painter can choose from red, blue, yellow, and orange for each floor in the building. Further he decides that each floor should have a different colour. In how many ways can he choose the colours to paint the building?

EXPLANATION

The painter can choose from four colors for the first floor.

The painter can choose from three colors for the second floor.

Total number of options = $4 \times 3 \times 2 = 24$

3. The principal of a school, who is an eccentric person, decides that all three floors should have a color that is a mixture of three colours that the painter has selected. In how many ways can the school be painted?

EXPLANATION

Now, this is different from the previous question in one key aspect. In the previous question, we select a color for the first floor, one for the second floor and one for the third floor.

Here we are going to select 3 colours out of 4 and mix them. Here, a selection of red, blue and green results in an identical outcome as that of a selection of blue, green and red. In other words, order does not matter.

The number of ways would be red–blue–yellow, red–blue–orange, red–yellow–orange, and blue–yellow–orange. There are only 4 options totally. Now, let us see if there is a method to arrive at this answer.

Let us look at this by enumerating some options. Look at the following six sequences. These are options that one could have chosen if one were painting each floor with a different colour (as outlined in the previous question).

First	Red	Red	Blue	Blue	Yellow	Yellow
Second	Blue	Yellow	Red	Yellow	Red	Blue
Third	Yellow	Blue	Yellow	Red	Blue	Red

In the case of selecting some set of three colours to get an overall blend, all the 6 options outlined above result in only one end outcome. So, the 4 × 3 × 2 options of the previous question result in $\frac{4 \times 3 \times 2}{6}$ options for this one. In other words, the number of ways of selecting 3 paints for a blend is $\frac{4 \times 3 \times 2}{3 \times 2 \times 1}$. Why is it 3 × 2 × 1 in the denominator? For every 3 × 2 ×1 outcomes for Question 2, there is only one end outcome when we consider the scenario outlined in Question 3.

What matters in this question is the combination of the colors and not the order. Hence considering the different possible arrangements of a selection is redundant here. And, we know that 3 distinct things can be rearranged in 3! ways; so we need to divide (4 × 3 × 2) by 3! to get the correct answer.

We are going to redefine the questions differently and create a general framework.

1. A school has to paint the 'r' floors in its building. The painter can choose from n different colours for each floor in the building. In how many ways can he choose the colours to paint the building?

Explanation

From n colours, select one for each floor, 'r' number of times. This can be done in n^r ways.

From n options, select 'r' such that order is important and repetition is allowed. This can be done in n^r ways.

2. A school has to paint the 'r' floors in its building. The painter can choose from 'n' different colours for each floor in the building. Further he decides that each floor should have a different colour. In how many ways can he choose the colours to paint the building?

Explanation

From n colours, select one distinct one for each of 'r' different floors.

From n options, select 'r' three such that order is important and repetition is not allowed. This can be done in n (n – 1) (n – 2) n – 3)….up to r terms. This can be re–written as $\frac{n!}{(n-r)!}$. This term is called ${}^nP_r$.

3. The principal of the school, who is an eccentric, decides that all 'r' floors should have the same colour and that colour should be a mixture of some 'r' of the 'n' colours available. In how many ways can the school be painted?

Explanation

From n colours, select r and then mix them together to get one blend to be used across the entire school

From n options, select 'r' such that order is not important and repetition is not allowed. This can be done in n (n – 1) (n – 2) (n – 3)….up to r terms/ (1 × 2 × 3 × …r). This can be re–written as $\frac{n!}{(n\ \ r)!r!}$. This term is called as ${}^nC_r$.

Order Important	$\mathbf{N^r}$	$\mathbf{{}^NP_r}$
Order not Important		$\mathbf{{}^NC_r}$
	Repetition Allowed	Repetition not Allowed

Also, the number of permutations can be seen as the number of combinations multiplied by the different arrangements possible for each combination. That is,

${}^nP_r = {}^nC_r \times r!$

Selecting in order = Selecting without order AND Rearranging them

Descriptive Examples

A football team plays a 5–a–side tournament. In all these tournaments, there should be 1 forward, 1 defender, 2–midfielders, and 1 goal keeper. 1 of the 5 should also be the captain of the team. Ram is the coach of team 'Samba' which has a squad of only 5 people.

Let us say the 5 people are A, B, C, D and E.

(a) In how many ways can Ram select a forward and a defender from this five?

EXPLANATION

Number of ways of selecting forward = 5.

Number of ways of selecting defender = 4.

Total number of outcomes = 5 × 4 = 20

The selections are as follows:

	BA	CA	DA	EA
AB		CB	DB	EB
AC	BC		DC	EC
AD	BD	CD		ED
AE	BE	CE	DE	

(b) In how many ways can Ram select a goal keeper and captain from this 5?

EXPLANATION

Number of ways of selecting goal keeper = 5.

Number of ways of selecting captain = 5.

Total number of outcomes = 5 × 5 = 25

The sections are as follows:

AA	BA	CA	DA	EA
AB	BB	CB	DB	EB
AC	BC	CC	DC	EC
AD	BD	CD	DD	ED
AE	BE	CE	DE	EE

(c) In how many ways can Ram select 2 midfielders from this 5?

EXPLANATION

Number of ways of selecting two players out of $5 = {}^5C_2 = \dfrac{5 \times 4}{2} = 10.$

The selections are as follows

AB				
AC	BC			
AD	BD	CD		
AE	BE	CE	DE	

Technically speaking, the three answers are ${}^5P_2$, 5^2 and ${}^5C_2$ respectively. The key difference between selecting forward and defender *vis–a–vis* two midfielders is that in the former order is important, in the latter order is not important.

PROBLEMS BASED ON DIGITS OF A NUMBER

1. How many three digit numbers exist? How many of these three–digit numbers comprise only even digits? In how many 3–digit numbers is the hundreds digit greater than the ten's place digit, which is greater than the units' place digit?

EXPLANATION

Three digit numbers range from 100 to 999. There are totally 900 such numbers. There is a simple framework for handling digits questions.

Let three digit number be 'abc'.

'a' can take values 1 to 9 {as the leading digit cannot be zero}.

'b' can take values 0 to 9.

'c' can take values 0 to 9.

Totally, there are 9 × 10 × 10= 900 possibilities.

Now, three digit number with even digits

Let the three–digit number be 'abc'.

'a' can take values 2, 4, 6 or 8 {as the leading digit cannot be zero}.

'b' can take values 0, 2, 4, 6 or 8.

'c' can take values 0, 2, 4, 6 or 8.

4 × 5 × 5 = 100 numbers

In how many 3–digit numbers is the hundreds digit greater than the ten's place digit, which is greater than the units' place digit?

The digits have to be from 0 to 9. Of these some three distinct digits can be selected in ${}^{10}C_3$ ways. For each such selection, exactly one order of the digits will have the digits arranged in the descending order. So, number of possibilities = ${}^{10}C_3 = 120$. We do not have to worry about the leading digit not being zero, as that possibility is anyway riled otu as a > b > c.

2. How many 4–digit numbers exist where all the digits are distinct?

Explanation

Let 4–digit number be 'abcd'.

'a' can take values from 1 to 9. **9 possibilities**

'b' can take values from 0 to 9 except 'a'. **9 possibilities**

'c' can take values from 0 to 9 except 'a' and 'b'. **8 possibilities**

'd' can take values from 0 to 9 except 'a', 'b' and 'c'. **7 possibilities**

Total number of outcomes = 9 × 9 × 8 × 7

PROBLEMS BASED ON REARRANGEMENT OF LETTERS OF A WORD

3. In how many ways can we rearrange the letters of the word 'MALE'? In how many ways can we rearrange the letters of the word 'ALPHA'? In how many ways can we rearrange the letters of the word 'LETTERS'?

Explanation

Number of ways of arranging letters of the word MALE = 4! = 24. (Think about the number of ways of arranging r distinct things).

Now, 'ALPHA' is tricky. If we had 5 distinct letters, the number of rearrangements would be 5!, but here we have two 'A's.

For a second, let us create new English alphabet with A_1 and A_2. Now the word A_1LPHA_2 can be rearranged in 5! ways. Now, in this 5! listings we would count A_1LPHA_2 and A_2LPHA_1, both of which are just ALPHA in regular English. Or, we are effectively double–counting when we count 5!. So, the total number of possibilities = $\frac{5!}{2}$. The formula is actually $\frac{5!}{2!}$. Whenever we have letters repeating we need to make this adjustment.

In how many ways can we rearrange the letters of the word 'LETTERS'? – $\frac{7!}{2!2!}$

4. In how many ways can we rearrange letters of the word 'POTATO' such that the two O's appear together? In how many ways can we rearrange letters of the word 'POTATO' such that the vowels appear together?

Explanation

The two O's appear together

Let us put these two O's in a box and call it X. Now, we are effectively rearranging the letters of the word PTATX. This can be done in $\frac{5!}{2!}$ ways.

Now, we need to count the ways when the vowels appear together. Let us put the three vowels together into a box and call it Y. We are effectively rearranging PTTY. This can be done in $\frac{4!}{2!}$ ways. However, in these $\frac{4!}{2!}$ ways, Y itself can take many forms. For instance, a word PTTY can be PTTAOO or PTTOAO or PTTOOA.

How many forms can Y take?

Y can take $\frac{3!}{2!} = 3$ forms.

So, total number of ways = $\frac{4!}{2!} \times \frac{3!}{2!} = 12 \times 3 = 36$ ways

PROBLEMS BASED ON DICE

5. In how many ways can we roll a die thrice such that all three outcomes are different? In how many ways can we roll a die thrice such that at least two throws are the same?

Explanation

Dice questions have a similar framework to digits question. When a die is thrown thrice, we can take the outcomes to be 'a', 'b', 'c'. There are two simple differences vis-a-vis digits questions.

(i) a, b, c can take only values from 1 to 6. In digits we have to worry about 0, 7, 8 and 9 as well

(ii) There are no constraints regarding the leading die. All throws have the same number of options.

So, in many ways, dice questions are simpler versions of digits questions.

In how many ways can we roll a die thrice such that all three throws show different numbers?

Let the throws be 'a', 'b', 'c'.

'a' can take 6 options – 1 to 6

'b' can take 5 options – 1 to 6 except 'a'

'c' can take 4 options – 1 to 6 except 'a' and 'b'

Total number of outcomes = $6 \times 5 \times 4 = 120$

In how many ways can we roll a die thrice such that at least two throws show the same number?

We can have either two throws same or all three same. There are 6 ways in which all three can be same — (1, 1, 1), (2, 2, 2), (3, 3, 3), (4, 4, 4), (5, 5, 5), (6, 6, 6).

Now, two throws can be same in three different forms a = b, b = c or c = a

a = b –> Number of outcomes = $6 \times 1 \times 5$. 'a' can take values from 1 to 6. b should be equal to 'a' and c can take 5 values – 1 to 6 except 'a'

b = c there are 30 possibilities, and

c = a there are 30 possibilities

Total number of options = 6 + 30 + 30 + 30 = 96

6. In how many ways can we roll a die twice such that the sum of the numbers on the two throws is an even number less than 8?

EXPLANATION

Sum can be 2, 4 or 6

2 can happen in one way: 1 + 1

4 can happen in 3 ways: 1 + 3, 2 + 2, 3 + 1

6 can happen in 5 ways: 1 + 5, 2 + 4, 3 + 3, 4 + 2, 5 + 1

Totally, there are 1 + 3 + 5 = 9 ways.

PROBLEMS BASED ON COIN TOSSES

7. When a coin is tossed three times, how many ways can exactly one head show up? When a coin is tossed coin 5 times, in how many ways can exactly 3 heads show up? When a coin is tossed 5 times, in how many ways can do utmost 3 heads show up?

EXPLANATION

Three coins are tossed, options with one head are HTT, THT and TTH. **3 ways**

5 coins are tossed, three heads can be obtained as HHHTT, HTHTH, TTHHH, …etc. Obviously this is far tougher to enumerate.

We can think of this differently. All the versions are nothing but rearrangements of HHHTT. This can be done in $\frac{5!}{3!2!}$ ways.

Coins questions are common, so it helps to look at them from another framework also.

Let us assume the outcomes of the 5 coin tosses are written down in 5 slots

___ ___ ___ ___ ___

Now, suppose, we select the slots that are heads and list them down.

So, a HHHTT would correspond to 123.

HTHTH would be 135.

TTHHH would be 345.

The list of all possible selections is nothing but the number of ways of selecting 3 slots out of 5. This can be done in ${}^5C_3$ ways, or, 10 ways.

Number of ways of getting exactly r heads when n coins are tossed = ${}^nC_r$

In how many ways can we toss a coin 5 times such that there are **utmost** 3 heads?

Utmost 3 heads => Maximum of three heads

Number of ways of having 3 heads = ${}^5C_3 = 10$

Number of ways of having 2 heads = ${}^5C_2 = 10$

Number of ways of having 1 head = ${}^5C_1 = 5$

Number of ways of having 0 heads = ${}^5C_0 = 1$

Utmost 3 heads = 10 + 10 + 5 + 1 = 26 ways.

8. When a coin is tossed 6 times, in how many ways do exactly 4 heads show up? Exactly 2 tails? When a coin is tossed 6 times, what is the total number of outcomes possible?

EXPLANATION

Coin tossed 6 times, number of ways of getting 4 heads = ${}^6C_4 = 15$

Exactly two tails = ${}^6C_2 = 15$

Every outcome where there are 4 heads corresponds to an outcome where there are 2 tails. So, we are effectively counting the same set of outcomes in both scenarios.

In other words ${}^nC_r = {}^nC_{n-r}$

Total number of outcomes = ${}^6C_0 + {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6 = 1 + 6 + 15 + 20 + 15 + 6 + 1 = 64$.

This 64 is also 2^6. When a coin is tossed once, there are 2 possible outcomes. When it is tossed 6 times there will be $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$ outcomes.

Therefore, ${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots. + {}^nC_n = 2^n$

GURU GYAN

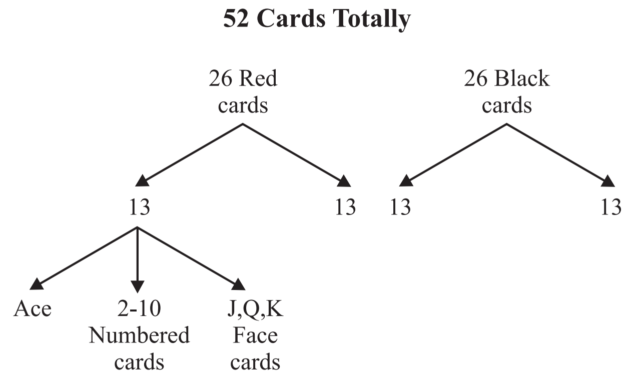
{This is also seen in the topic 'Binomial Theorem'. But never hurts to reiterate!}

PROBLEMS BASED ON CARD PACKS

Quesitons based on card pack are very straightforward. We need to know exactly what lies inside a card pack. Once we know this, everything else falls in place.

A card pack has 52 cards - 26 in red and 26 in black. There are 4 suits totally, 2 of each colour. Each suit comprises 13 cards - an Ace, numbers 2 to 10, and Jack, Queen and King.

The cards with numbers 2 to 10 are called numbered cards. Cards with J, Q, K are called Face cards as there is a face printed on them.



9. In how many ways can we select 4 cards from a card pack such that all are face cards?

EXPLANATION

There are 12 face cards in a pack. Number of ways of selecting 4 out of these = ${}^{12}C_4$

10. In how many ways can we select 3 cards from a card pack such that none are black numbered cards?

EXPLANATION

There are 18 black numbered cards. If we select 3 cards and none of these are black numbered cards, then all of these must be from the remaining 34. Number of ways of selecting 3 cards from 34 is ${}^{34}C_3$

11. In how many ways can we select 5 cards from a card pack such that we select at least 1 card from each suit?

EXPLANATION

We should select 1 card each from 3 of the suits and 2 from the fourth. The suit that contributes the additional card can be selected in ${}^4C_1$ ways.

So, total number of outcomes = ${}^4C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_2$

CIRCULAR ARRANGEMENT

Let us say there are n people to be seated around a circular table. In how many ways can this happen? If we think about this as n slots, where n things are to fit in, the answer would be n!. But, what we miss out here is the fact that if every object moves one step to the right/left then this would not be a different arrangement. In fact any rotation of one arrangement is not giving us another arrangement. So, how do we think about this? Let us fix one person's position. Then, we have the remaining (n–1) persons who can be arranged in (n–1)! ways.

12. In how many ways can 6 people be arranged around a circle? In how many ways can 6 people be arranged around a circle if A and B should never sit together?

EXPLANATION

Number of ways of arranging n people around a circle = (n –1)!. So, the number of ways of arranging 6 people around a table = (6 –1)! = 5! = 120.

Now, A and B should never sit together. Let us calculate all the possibilities where A and B do sit together. Now, A and B can be together called X. Number of ways of arranging 5 around a circle = 4!. Now, AB can be sitting such that A is to the left of B or B is to the left of A. So, total number of options = 4! × 2.

Number of options where A and B do not sit adjacent to each other = 5! – 2 × 4! = 120 – 48 = 72.

SELECTING ONE OR MORE FROM A SET

13. If there are 4 books and 3 CDs on a table, in how many ways can we select at least one item from the table?

EXPLANATION

Each article has 2 options, either you can select it or you can skip it.

Total no of option = 2^7. Within this 2^7 possibilities, there is one possibility that we skip ALL the items. Since we need to select at least least one item, we need to subtract this possibility. Total number of ways = $2^7 - 1 = 127$.

14. If there are 4 identical copies of a book and 3 identical copies of a CD on a table, in how many ways can we select at least one item from the table?

EXPLANATION

We can select either 0, 1, 2, 3 or 4 books. Similarly, we can select 0, 1, 2 or 3 CDs. So, total number of options = 5 × 4 = 20 ways. Of these one will include the option of not selecting any of the things. So, total number of possible outcomes = 4 × 5 – 1 = 19.

Note that here we do not worry about WHICH CD or book we are selecting. Since the CDs and books are identical, only the number of CDs/books matters.

DISTRIBUTION INTO GROUPS

15. In how many ways can we split 8 different objects into groups of 5 and 3? In how many ways can we split this into groups of 4, 3 and 1? In how many ways can we split this into groups of 4, 2, and 2?

EXPLANATION

Number of ways of splitting 8 different objects into groups of 5 and 3 = $^8C_5$: Select 5 objects, the remaining 3 form the second group automatically. You should also note that it is same as selecting 3 objects from 8, $^8C_3$. $^8C_5 = {}^8C_3$.

8 different objects into groups of 4, 3 and 1: First select 4 objects – this can be done in $^8C_4$ ways. Then select 3 out of the remaining 4 – this can be done in $^4C_3$ ways.

Total number of ways

$$= {}^8C_4 \times {}^4C_3 = \frac{8!}{4!4!} \times \frac{4!}{3!\ 1!} = \frac{8!}{4!3!1!}$$

So, it follows that p + q + r objects can be split into groups of p, q and r in $\frac{(p+q+r)!}{p!q!r!}$ ways as long as p, q, r are distinct.

8 different objects into groups of 4, 2 and 2: Now, we need to treat this differently. From 8 objects, number of ways of selecting 4 is $^8C_4$. Post this, number of ways of selecting 2 out of 4 would be $^4C_2$. But $^8C_4 \times {}^4C_2$ would overstate the number. In the final $^4C_2$, we calculate the number of ways of selecting 2 objects from 4, with the assumption that the remaining 2 would form the other group.

However, let us say we need to select two out of A, B, C, and D. For every selection of AB in the Group 1 and CD for Group 2, we could have an exact mirror selection CD in Group 1 and AB in Group 2. So, the correct answer should be $\frac{^8C_4 \times {}^4C_2}{2!}$. Every time we split and allocate into groups of same size, we need to be careful.

DISTRIBUTION INTO GROUPS – VARIANTS

16. In how many ways can 6 identical toys be placed in 3 identical boxes such that no box is empty?

Explanation

This is simple. Since the toys and boxes are identical, we just need to deal with a ways of splitting six into three natural numbers

$1 + 2 + 3 = 6$

$1 + 1 + 4 = 6$

$2 + 2 + 2 = 6$

These are the three ways in which it can be done.

17. In how many ways can 6 identical toys be placed in 3 distinct boxes such that no box is empty?

Explanation

Again, we are looking to solve for $a + b + c = 6$. But in this case, a (1, 2, 3) will be counted as a separate from (3, 2, 1) and (2, 1, 3).

(1, 2, 3) can be rearranged in $3! = 6$ ways

(1, 1, 4) can be rearranged in $^3C_1$ ways = 3 ways. We need to select which box has the set of 4 toys. The other two boxes will have 1 each. Alternatively, we are thinking about in how many ways we can rearrange 114. This can be done in $\frac{3!}{2!}$ ways.

(2, 2, 2) can be done in only one way.

So, totally we have $6 + 3 + 1 = 10$ ways.

Alternative Method

$a + b + c = 6$. Now, let us place six sticks in a row

| | | | | |

This question now becomes the equivalent of placing two '+' symbols somewhere between these sticks. For instance

| | + | | | + |, This would be the equivalent of 2 + 3 + 1. or, a = 2, b = 3, c = 1.

There are 5 slots between the sticks, out of which one has to select 2 for placing the '+'s.

The number of ways of doing this would be $^5C_2 = 10$. Bear in mind that this kind of calculation counts ordered triplets. (2, 3, 1) and (1, 2, 3) will both be counted as distinct possibilities. This is why we use this method for finding the number of ways of placing 6 identical objects in 3 **distinct** boxes.

So, the above question can also be phrased like this: In how many ways can we have ordered triplets of natural numbers (a, b, c) such that $a + b + c = 6$.

There is another version of this question with ordered triplets of whole numbers. Think about what adjustment needs to be done there.

18. In how many ways can 6 distinct toys be placed in 3 identical boxes such that no box is empty?

Explanation

First let us think of the distributions. The boxes can have

1, 2, 3: This can be done in $^6C_3 \times {}^3C_2$ ways. First select 3 out of 6, and then 2 out of the remaining 3. This is nothing but distributing 6 as 3, 2, 1 which can be done in $\frac{6!}{2! \times 3! \times 1!}$ ways

1, 1, 4: This can be done in $^6C_4$ ways. Once we select 4 out of 6, the other two go into one box each. Since the boxes are identical, we do not have to worry about selecting anything beyond the first set of 4 toys.

2, 2, 2: This looks like it could be $^6C_2 \times {}^4C_2$ ways. But this will carry some multiple counts. The idea we are using here is simple – select 2 out of 6 and then select 2 out of 4.

When we do this, a selection of AB, and then CD will get counted. This will get accounted as AB, CD, EF. However, we will also be counting a selection of CD, AB, EF, and EF, AB, CD. Since the boxes are identical, all these selections are effectively the same. So, number of ways would be $\frac{^6C_2 \times {}^4C_2}{3!}$

So, total number of ways of doing this would be 60 + 15 + 15 = 90 ways.

19. In how many ways can 6 distinct toys be placed in 3 distinct boxes such that no box is empty?

EXPLANATION

Again, let us start with the distributions.

Scenario I: (1, 2, 3): This can be done in $^6C_3 \times {}^3C_2 \times 3!$ ways. First select 3 out of 6, and then 2 out of the remaining 3. After we have done this, the toys can go into the three distinct boxes in 3! ways. **360 ways**

Scenario II: 1, 1, 4: This can be done in $^6C_4 \times 3!$ ways. Once we select 4 out of 6, the other two go get broken up as 1 and 1. Now, we have something akin to ABCD, E and F to be allotted into 3 distinct boxed. This can be done in 3! ways. **90 ways**

Scenario III: 2, 2, 2: This should be $^6C_2 \times {}^4C_2$ ways. The idea we are using here is simple – select 2 out of 6 for the first box and then select 2 out of 4 for the second box. **90 ways.**

Total number of ways = 360 +90 + 90 = **540 ways.**

Now, this question can be rephrased wonderfully like this:

How many onto functions can be defined from {a, b, c, d, e, f} to {1, 2, 3}?

You can solve the above question by thinking of all functions from the first set to the second and subtracting the non–onto functions from that. Needless to say, we would get the same answer.

GURU GYAN

Practically the same question can be asked in dramatically different contexts. It is important to 'pick' that 2 questions are just versions from the same template.

COUNTING AND NUMBER THEORY

20. How many factors of $2^7 \times 11^5 \times 7^4$ are perfect squares?

EXPLANATION

Any factor of $2^7 \times 11^5 \times 7^4$ will be of the form $2^a \times 11^b \times 7^c$.

$a \le 7$

$b \le 5$

$c \le 4$

Any perfect square's prime factorisation will have all the powers as even numbers. So, a can take values 0, 2, 4, 6; b can take values 0, 2 or 4; and c can take values 0, 2 or 4.

Number of factors that are perfect squares are $4 \times 3 \times 3 = 36$

21. All numbers from 1 to 250 (in decimal system) are written in base 7 and base 8 systems. How many of the numbers will have a non–zero units digit in both base 8 and base 7 notations?

EXPLANATION

A number when written in base 8, if it ends in 0, should be a multiple of 8. Likewise for base 7. So, effectively this question becomes — How many natural numbers exist less than 251 that are multiples of neither 7 nor 8.

Let us first find out numbers that are multiples of either 7 or 8.

Multiples of 7 — 7, 14, 21, 28,.......245... 35 numbers in this list.

Multiples of 8 — 8, 16, 24, 32,.......248... 31 numbers in this list.

Some numbers will be multiples of 7 and 8.

Multiples of 56 — 56, 112, 168, 224… 4 numbers in this list

Number of numbers that are multiples of 7 or 8 = 35 + 31 – 4 = 62

Number of numbers that are multiples of neither 7 nor 8 = 250 – 62 = 188

22. How many natural numbers less than 10000 exist such that sum of their digits is 6?

EXPLANATION

We are considering all numbers up to 9999.

All numbers of the form ABCD such that a, b, c, d can take values from 0 to 9.

a + b + c + d = 6 where a, b, c, d are all whole numbers.

Now, a, b, c, d can take value 0. Let us simplify this as p = a + 1, q = b + 1, r = c + 1, s = d + 1.

Now, p + q + r + s = 10. p, q, r, s cannot be zero.

Number of solutions for the above equation is $^9C_3$.

Bear in mind that p, q, r, s can all never be greater than 10. In this case, as the total adds up to 10, we have little to worry about. If the sum were greater than 10, it could become far more complex.

23. How many numbers are factors of 24^{20} but not of 24^{15}?

EXPLANATION

$24^{20} = (2^3 3)^{20} = 2^{60}3^{20}$. Number of factors of this number = 61 × 21 = 1281

$24^{15} = 2^{45}3^{15}$ Number of factors = 46 × 16 = 736

Factors of 24^{15} will be a subset of factors of 24^{20}. So, the number of numbers that are factors of 24^{20} but not of 24^{15} is nothing but 61 × 21 – 46 × 16 = 1281 – 736 = 545.

24. How many natural numbers less than 10^4 exist that are perfect squares but not perfect cubes?

EXPLANATION

Number of perfect squares less than 10000 = 99. 10000 = 100^2; so till 99^2 will be less than 10000. So, there are 99 perfect squares less than 10000? From these some numbers that are also perfect cubes have to be eliminated.

So, we are looking for numbers that are perfect squares and perfect cubes. Or, we are looking for powers of 6.

1^6, 2^6, 3^6, 4^6 are all less than 10000, but 5^6 is greater than 10000. So, there are only 4 powers of 6.

So, out of 99, we need to subtract 4 possibilities. Or, there are 95 different natural numbers that will be perfect squares but not perfect cubes.

EXERCISE PROBLEMS

1. In how many ways can the letters of the word 'MALAYALAM' be rearranged? In how many of the words would the A's appear together? In how many of the words are the consonants together?

EXPLANATION

Rearrangements of MALAYALAM = $\frac{9!}{4!2!2!}$

Rearrangements where the A's appear together: Put 4 A's together into X and rearrange MLYLMX. This can be rearranged in $\frac{6!}{2!2!}$

Rearrangements where the consonants appear together: Put consonants together into Z. The word will be AAAAZ. This can be rearranged in $\frac{5!}{4!}$ ways.

Now, Z is MMLLY. Z can take $\frac{5!}{2!2!}$ ways.

So, total number of arrangements = $\frac{5!}{4!} \times \frac{5!}{2!2!}$

2. In Octaworld, everything is written in base 8 form. Ram's 1050 page tome written in the real world is translated to Octa–speak and reprinted in Octa–world. In the reprint, each page number is written in base 8 form. How many times will the digit 4 be printed on the page numbers?

EXPLANATION

$(1050)_{10} = (2032)_8$.

So, we need to see how many times the digit 4 gets printed from 1 to 2032 in base 8.

Let us consider a number of the form $(a\,b\,c)_8$ where a, b c take all digits from 0 to 7. Essentially, in decimal equivalent, we are considering all numbers from 1 to 511.

When c = 4, a can take all values from 0 to 7 and b can take all values from 0 to 7. So, 4 gets printed at c 64 times.

When b = 4, a can take all values from 0 to 7 and c can take all values from 0 to 7. So, 4 gets printed at b 64 times.

When a = 4, b can take all values from 0 to 7 and c can take all values from 0 to 7. So, 4 gets printed at a 64 times.

So, totally 4 gets printed 64 × 3 = 192 times from $(1)_8$ to $(777)_8$

From $(1000)_8$ to $(1777)_8$, 4 would get printed 192 times.

So, up to $(2000)_8$, the digit 4 gets printed 192 × 2 = 384 times.

We need to account for all base 8 numbers from $(2001)_8$ to $(2032)_8$. In these 26 numbers, the digit 4 gets printed thrice $(2004)_8$, $(2014)_8$, $(2024)_8$. So, the digit 4 gets printed 384 + 3 = 387 times.

3. When a die is thrown twice, in how many outcomes will the product of the two throws be 12?

EXPLANATION

12 can be formed from 3 × 4 or 2 × 6 {1 × 12 is not possible with die}.

This can happen in 2 + 2 = 4 ways.

4. How many words exist that have exactly 5 distinct consonants and 2 vowels?

EXPLANATION

Scenario I: 5 distinct consonants and 2 distinct vowels – Number of words = ${}^{21}C_5 \times {}^5C_2 \times 7!$

Scenario II: 5 distinct consonants and 1 vowel appearing twice – Number of words = ${}^{21}C_5 \times {}^5C_1 \times \frac{7!}{2!}$

5. When a coin is tossed 6 times, in how many outcomes will there be more heads than tails?

EXPLANATION

We should have more heads than tails => There should be 4 heads or 5 heads or all 6 heads.

Number of ways = ${}^6C_4 + {}^6C_5 + {}^6C_6 = 15 + 6 + 1 = 22$

6. In how many ways can we pick 4 cards from a card pack such that there are no Aces selected and there are more face cards than numbered cards?

EXPLANATION

Scenario I: 3 face cards and 1 numbered card: ${}^{12}C_3 \times {}^{36}C_1$

Scenario II: 4 face cards and 0 numbered cards: ${}^{12}C_4$

Therefore, total number of ways is ${}^{12}C_3 \times {}^{36}C_1 + {}^{12}C_4$

7. On a table, there are 4 identical copies of a book and 3 CDs. In how many ways can we pick at least one book and at least one CD from the table?

EXPLANATION

There are 4 identical copies of a book. One can pick either 0, 1, 2, 3, or all 4 of these – 5 different options. We need to pick at least one book. So, we have only 4 options – 1, 2, 3, or 4 books being picked

There are 3 different CDs. Each CD can be either picked or not picked. So, total number of options = 2^3. Of these there is one option where no CD is picked. We need to exclude that option. So, number of possibilities = $2^3 - 1$

Total number of outcomes = $4 \times (2^3 - 1) = 4 \times 7 = 28$

8. What is sum of all 4-digit numbers formed by rearranging the digits of the number 2235?

EXPLANATION

Number of rearrangements of 2235 = $\frac{4!}{2!} = 12$. So, we need to add these 12 numbers. Let us consider the units' digit of these 12 numbers.

The units digit will be the one of the digits 2, 3, or 5. If the last digit were 3, the first 3 digits should be some rearrangement of 2, 2, and 5. So, there are $\frac{3!}{2!}$ Such numbers. Or, 3 such numbers.

Similarly there are three numbers with 5 as the units digit.

If the last digit were 2, the first 3 digits should be some rearrangement of 2, 3, and 5. So, there are 3! such numbers, or, 6 such numbers.

So, the units digit will be 2 for six numbers, 3 for three numbers, and 5 for three numbers. Sum of all these unit digits will be $2 \times 6 + 3 \times 3 + 5 \times 3 = 12 + 9 + 15 = 36$.

Sum of all the tens digits will be 36. Sum of all the digits in the hundreds' place will be 36. Sum of all the digits in the thousands' place is 36.

So, sum of all the 4–digit numbers will be $36 \times 1111 = 39996$.

9. When a die is thrown twice, in how many ways can we have the sum of numbers to be less than 8?

EXPLANATION

Sum of the numbers seen in the two throws can be 2, 3, 4, 5, 6 or 7.

Sum of the digits = 2: This can only be 1 + 1. **One way**

Sum of the digits = 3: This can be 1 + 2 or 2 + 1. **2 ways**

Sum of the digits = 4: 1 + 3, 3 + 1, 2 + 2. **3 ways**

Sum of the digits = 5: 1 + 4, 4 + 1, 3 + 2, 2 + 3. **4 ways**

Sum of the digits = 6: 1 + 5, 5 + 1, 2 + 4, 4 + 2, 3 + 3. **5 ways**

Sum of the digits = 7: 1 + 6, 6 + 1, 2 + 5, 5 + 2, 3 + 4, 4 + 3. **6 ways**

Sum of the numbers in the two throws can be less than 8 in 1 + 2 + 3 + 4 + 5 + 6 = **21 ways.**

We notice a very simple pattern here. Try the sum of the numbers all the way to 12 and see the rest of the pattern also.

10. Set P has elements {1, 2, 3.….10}. How many non–empty subsets of P have the product of their elements as not a multiple of 3?

EXPLANATION

Total number of subsets = 2^{10}

For choosing any subset, each element can either be part of the subset or not part of the subset. So, for each element, there are two options. So, with 10 elements in the set, we can create 2^{10} subsets. We should bear in mind that this 2^{10} includes the 2 improper subsets as well. The whole set P and the null set are included in this 2^{10}.

Subsets whose product is not a multiple of 3 = Subsets of the set {1, 2, 4, 5, 7, 8, 10} = 2^7. This includes the empty subset also. So, the correct answer should be $2^7 - 1$

11. A, B, C, D, E are doctors, P, Q, R, S, are engineers. In how many ways can we select a committee of 5 that has more engineers than doctors?

EXPLANATION

Two scenarios are possible.

3 engineers and 2 doctors: ${}^4C_3 \times {}^5C_2 = 4 \times 10 = 40$

4 engineers and 1 doctor : ${}^4C_4 \times {}^5C_1 = 1 \times 5 = 5$

Total number of possibilities = 40 + 5 = 45

12. From a card pack of 52, in how many ways can we pick a sequence of 4 cards such that they are in order and from different suits? Consider Ace to be the card following King in each suit. So, Ace can be taken to precede '2' and succeed 'King'. So, JQKA would be a sequence, so would be A234. However, QKA2 is not a sequence.

EXPLANATION

4 cards in order can be A234, 2345, ….JQKA. 11 different possibilities

For a given set of four cards, say 2345, they can be from 4 different suits in 4! ways.

So, total number of possibilities = 11 × 4! = 264.

CAT Level Questions

1. In how many ways can letters the word ATTITUDE be rearranged such that no two T's are adjacent to each other?

(a) 6720 (b) 2400
(c) 4320 (d) 1800

2. In how many rearrangements of the word SLEEPLESS will no two S's appear together?

(a) 2100 (b) 12600
(c) 1050 (d) 4200

3. How many numbers of up to 5 digits can be created using the digits 1, 2 and 3 that are multiples of 12?

(a) 18 (b) 23
(c) 27 (d) 26

4. How many numbers of up to 5 digits can be created using the digits 1, 2, 3, 5 each at least once such that they are multiples of 15?

(a) 24 (b) 18
(c) 15 (d) 12

5. How many 4–digit numbers with distinct digits exist, product of whose digits is a non–zero multiple of 9?

(a) 1008 (b) 1334
(c) 2448 (d) 1704

6. From 4 Doctors, 3 Engineers and 5 Scientists, in how many ways can we create a committee of 6 to 8 people that has more Scientists than Engineers, more Engineers than Doctors, and at least one Doctor?

(a) 222 (b) 212
(c) 232 (d) 202

7. A flag is formed with 4 vertical bands. If we can choose from colors blue, green, and yellow and no two adjacent bands should have the same color, how many different flags can be created?

(a) 6 (b) 18
(c) 24 (d) 30

8. An equation $ax^2 + 8x + c = 0$ has two distinct real roots. If a, c are positive integers less than 11, how many values can (a, c) take?

(a) 43 (b) 34
(c) 71 (d) 35

9. $2a + 5b = 103$. How many pairs of positive integer values can a, b take such that $a > b$?

(a) 7 (b) 9
(c) 14 (d) 15

10. Using the vertices of a regular hexagon as vertices, triangles of how many different areas can be formed?

(a) 3 (b) 4
(c) 8 (d) 20

11. Diagonals of a square ABCD of side 35 cms intersect at O. A rhombus of perimeter 52 exists such that P lies on AO, Q on BO, R on CO and S on DO. What is the maximum number of circles with integer radii that can be drawn with center O such that the rhombus is inside the circle and the circle is inside the square?

(a) 9 (b) 8
(c) 10 (d) 7

12. The diagonals of Hexagon intersect at n distimct points inside the hexagon. What is the maximum value n can take?

(a) 12 (b) 20
(c) 15 (d) 18

13. Consider a circle of radius 6 cms. What is the maximum number of chords of length 6 cms that can be drawn in the circle such that no two chords intersect or have points of contact?

(a) 4 (b) 5
(c) 6 (d) 8

14. $x(x - 3)(x + 2) < 200$, and x is an integer such that $|x| < 20$. How many different values can n take?

(a) 19 (b) 20
(c) 26 (d) 27

15. In how many ways can we pick three cards from a card pack such that they form a sequence of consecutive cards? Not all cards belong to the same suit, and nor do all cards belong to distinct suits? Consider Ace to be the card following King in each suit. So, Ace can be taken to precede '2' and succeed 'King'. So, QKA would be a sequence, so would be A23. However, KA2 is not a sequence.

(a) 216 (b) 864
(c) 432 (d) 144

16. Given that $|k| < 15$, how many integer values can k take if the equation $x^2 - 6|x| + k = 0$ has exactly 2 real roots?

(a) 15 (b) 14
(c) 16 (d) 13

17. In how many ways can 6 boys be accommodated in 4 rooms such that no room is empty and all boys are accommodated?

(a) 480 (b) 1080
(c) 1560 (d) 1920

18. In how many ways can 4 boys and 4 girls be made to sit around a circular table if no two boys sit adjacent to each other?

(a) 576 (b) 288
(c) 144 (d) 36

19. John extracts three letters from the word 'ACCEDE' and makes words out of them. How many different words can he generate?

(a) 24 (b) 32
(c) 18 (d) 42

20. Joseph extracts three letters from the word 'RENEGED' and makes words out of these 3 letters. How many such words can he generate?

(a) 72 (b) 60
(c) 61 (d) 73

21. If we listed all numbers from 100 to 10,000, how many times would the digit 3 be printed?

(a) 3980
(b) 3700
(c) 3840
(d) 3780

22. How many odd numbers with distinct digits can be created using the digits 1, 2, 3, 4, 5 and 6?

(a) 975 (b) 960
(c) 978 (d) 986

23. How many 5–digit numbers with distinct digits can be created with digits 1, 2, 3, 4, 5, 6 such that the number is multiple of 12?

(a) 36 (b) 60
(c) 24 (d) 72

24. How many 6 letter words with distinct letters exist that have more vowels than consonants?

(a) 771120 (b) 668240
(c) 846820 (d) 108120

25. How many 4 letters words can be created with more consonants than vowels?

(a) ${}^{21}C_1 \times {}^{5}C_1 \times \frac{4!}{3!}$

(b) $21^4 + {}^{21}C_3 \times {}^{5}C_1 \times 4! + {}^{21}C_2 \times {}^{2}C_1 \times {}^{5}C_1 \times \frac{4!}{2!} + {}^{21}C_1 \times {}^{5}C_1 \times \frac{4!}{3!}$

(c) $21^4 + {}^{21}C_3 \times {}^{5}C_1 \times 4! + {}^{21}C_2 \times {}^{2}C_1 \times {}^{5}C_1 \times \frac{4!}{2!}$

(d) $21^4 + {}^{21}C_3 \times {}^{5}C_1 \times 4! + {}^{21}C_2 \times {}^{2}C_1 \times {}^{5}C_1 \times \frac{4!}{2!} + {}^{21}C_1 \times {}^{5}C_1 \times \frac{4!}{3!} + {}^{21}C_2 \times {}^{2}C_2 \times {}^{5}C_1 \times \frac{4!}{2!}$

26. All the rearrangements of the word "DEMAND" are written without including any word that has 2 D's appearing together. If these are arranged alphabetically, what would be the rank of "DEMAND"?

(a) 36 (b) 74
(c) 42 (d) 86

27. If the letters of the word SLEEPLESS were arranged alphabetically, what rank would 'SLEEPLESS' hold?

(a) 4003 (b) 4018
(c) 4015 (d) 3991

28. If all words with 2 distinct vowels and 3 distinct consonants were listed alphabetically, what would be the rank of "ACDEF'?

(a) 4716 (b) 4720
(c) 4718 (d) 4717

29. When a die is thrown 4 times, in how many ways can we have at least one digit appearing exactly twice?

(a) 720 (b) 640
(c) 810 (d) 1620

30. When a die is rolled 3 times, how many possibilities exist such that the outome of each throw is a number at least as high as that of the preceding throw?

(a) 56 (b) 55
(c) 58 (d) 52

31. When a die is rolled thrice, in how many outcomes will we have the product of the throws and sum of the throws to be even numbers?

(a) 108 (b) 81
(c) 54 (d) 144

32. A string of n consecutive even natural numbers has 23 multiples of 14 in it. How many of the following could be true?

1. n = 157
2. Of these n numbers, exactly 80 are multiples of 4.
3. Of these n numbers, more than 50 are multiples of 3.

(a) 0 (b) 1
(c) 2 (d) 3

33. The product of the digits of a 5–digit number is 1800. How many such numbers are possible?

(a) 180 (b) 300
(c) 120 (d) 240

34. The sum of three distinct positive integers is 16. How many values can a x b x c take?

(a) 7 (b) 11
(c) 14 (d) 13

35. The product of 3 distinct positive numbers is 120. How many such sets are possible?

(a) 13 (b) 14
(c) 12 (d) 11

36. Set A has element {a, b, c, d, e} set B = {1, 2, 3}. How many onto functions exist from set A to set B such that f(a) = 1.

(a) 150 (b) 14
(c) 50 (d) 16

37. Set A has elements {a, b, c, d, e, f}. How many subsets of A have element 'a', do not have element b and have an even numbers of elements?

(a) 4 (b) 6
(c) 8 (d) 2

38. How many rearrangements of the word EDUCATION are there where no two consonants are adjacent to each other?

(a) 43200 (b) 21600
(c) 2880 (d) 1800

39. Consider 24 points of which 7 lie on a straight line and 8 are on a different straight line. No other set of 3 points are collinear. How many triangles can be considered out of these 24 points?

(a) 1849 (b) 1933
(c) 1393 (d) 1429

40. How many numbers with distinct digits are possible, the product of whose digits is 28?

(a) 6 (b) 4
(c) 8 (d) 12

SOLUTIONS

1. In how many ways can letters of the word ATTITUDE be rearranged such that no two T's are adjacent to each other?

CAT LEVEL 2

SOLUTION

ATTITUDE has 8 letters, of which 3 are T's

Now, Let us place the letters that are not Ts on a straight line. We have AIUDE. These can be arranged in 5! Ways. Now let us create slots between these letters to place the Ts in. In order to ensure that no two T's are adjacent to each other, let us create exactly one slot between any two letters.

A __ I __ U __ D __ E.

Additionally, let us add one slot at the beginning and end as well as the T's can go there also

__A __ I __ U __ D __ E__

Now, out of these 6 slots, some 3 can be T. That can be selected in ${}^6C_3$ ways.

So, total number of words = $5! \times {}^6C_3 = 2400$

Answer choice (b)

2. In how many rearrangements of the word SLEEPLESS will no two S's appear together?

CAT LEVEL 1

SOLUTION

Let us aggregate the other letters LEEPLE and arrange these. These can be arranged in $\frac{6!}{2!3!}$ ways.

Now, let us create slots in between and before/after these letters, where S can potentially appear.

__ L __ E __ E __ P __ L __ E__. Out of these 7 slots, 3 should be taken up by S's. This can be done in 7C3 ways.

So, total number of rearrangements = $\frac{6!}{2!3!} \times {}^7C_3$ = $60 \times 35 = 2100$

Answer choice (a)

3. How many numbers of up to 5 digits can be created using the digits 1, 2 and 3 that are multiples of 12?

CAT LEVEL 2

SOLUTION

For a number to be a multiple of 12, it has to be a multiple of 3 and of 4. So, the last two digits have to be a multiple of 4 and the sum of digits should be a multiple of 3.

We need to break this down by number of digits.

2–digit number: Only possibility 12.

3–digit numbers: These can end in 12 or 32. If it ends in 12, the sum of these two digits is 3, the only value the first digit can take is 3. Similarly if it ends in 32, the only value the first digit can take is 1. So, two 3–digit numbers are possible 312 and 132.

4–digit numbers: Again, these can in 12 or 32.

Scenario I: Ending in 12. Sum of these two digits = 3. First two digits can be 12 or 21 or 33.

Scenario II: Ending in 32. Sum of these two digits = 5. First two digits can be 22, 13 or 31.

So, possible 4–digit numbers are 1212, 2112, 3312, 2232, 1332, 3132.

5–digit numbers: Again, these can end in 12 or 32.

Scenario I: Ending in 12. Sum of these two digits = 3. First three digits can be 111, 222, 333, 123 (6 rearrangements 132, 213, 231, 312, 321) – 9 possibilities

Scenario II: Ending in 32. Sum of these two digits = 5. First three digits can add up to 4 or 7.

- First 3–digits adding up to 4: 112, 121, 211. 3 possibilities

- First 3–digits adding up to 7: 223, 232, 322, 133, 313, 331. 6 possibilities
- 9 possibilities overall

Total number of 5–digit numbers = 9 + 9 = 18

Overall, total number of possibilities = 1 + 2 + 6 + 18 = 27 numbers.

Answer choice (c)

4. How many numbers of up to 5 digits can be created using the digits 1, 2, 3, 5 each at least once such that they are multiples of 15?

CAT LEVEL 2

SOLUTION

For a number to be a multiple of 15, it has to be a multiple of 3 and of 5. So, the last digit has to be 5 and the sum of digits should be a multiple of 3.

We can have either 4–digit or 5–digit numbers. If we have a 4–digit number, sum of the digits will be 1 + 2 + 3 + 5 = 11. No 4–digit number formed with digits 1, 2, 3, 5 exactly once can be a multiple of 3. So, there is no possible 4–digit number.

Now, in any 5 digit number, we will have 1, 2, 3, 5 once and one of these 4 digits repeating once. 1 + 2 + 3 + 5 = 11. So, the digit that repeats in order for the number to be a multiple of 3 has to be 1. In this instance, sum of the digits will be 12 and this is the only possibility.

So, any 5–digit number has to have the digits 1, 1, 2, 3, 5. For the number to be a multiple of 5, it has to end in 5.

So, number should be of the form __ __ __ __ 5, with the first 4 slots taken up by 1, 1, 2, 3. These can be rearranged in $\frac{4!}{2!} = 12$ ways.

There are 12 possibilities overall.

Answer choice (d)

5. How many 4–digit numbers with distinct digits exist product of whose digits is a non–zero multiple of 9?

CAT LEVEL 2

SOLUTION

For the product to be zero, one of the digits has to be zero. So, if the product is non–zero, no digit can be zero.

For the product to be a multiple of 9, one of the digits can be 9, or we could have 3 and 6 as two digits of the number.

Scenario I: One of the digits being equal to 9: 9 can be either in the 1000's place, or 100's place, or 10's place or units place. Now, with 9 in the 1000's place, we will have 8 × 7 × 6 numbers totally. So, overall there will be 8 × 7 × 6 × 4 numbers possible. 56 × 24 = 1344 numbers

Scenario II: Two of the digits being equal to 3 and 6: Now, in this list we should exclude all numbers that have a 9 as we would have already accounted for these. So, we need to count all possibilities where two digits being 3 and 6, other two selected from 1, 2, 4, 5, 7, 8. No of ways of selecting 2 digits out of 6 = $^6C_2$. Number of ways of rearranging = 4!.

So, number of numbers = $^6C_2 \times 4! = 15 \times 24 = 360$

Total number of possible numbers = 1344 + 360 = 1704.

Answer choice (d)

6. From 4 doctors, 3 engineers and 5 scientists, in how many ways can we create a committee of 6 to 8 people that has more scientists than engineers, more engineers than doctors, and at least one doctor?

CAT LEVEL 1

SOLUTION

Three scenarios are possible – the committee can have 6, 7 or 8 people

Scenario I: 6–member committee. 3 scientists, 2 engineers and 1 doctor. Number of ways = $^5C_3 \times {}^3C_2 \times {}^4C_1 = 10 \times 3 \times 4 = 120$

Scenario II: 7–member committee.

- 4 scientists, 2 engineers and 1 doctor: Number of ways = ${}^5C_4 \times {}^3C_2 \times {}^4C_1 = 5 \times 3 \times 4 = 60$

Scenario III: 8–member committee.

- 5 scientists, 2 engineers and 1 doctor: Number of ways = ${}^5C_5 \times {}^3C_2 \times {}^4C_1 = 1 \times 3 \times 4 = 12$
- 4 scientists, 3 engineers and 1 doctor: Number of ways = ${}^5C_4 \times {}^3C_3 \times {}^4C_1 = 5 \times 1 \times 4 = 20$ ways

Total number of possibilities = 120 + 60 + 12 + 20 = 212.

Answer choice (b)

7. A flag is formed with 4 vertical bands. If we can choose from colors blue, green, and yellow and no two adjacent bands should have the same color, how many different flags can be created?

CAT LEVEL 2

Solution

Scenario I: 2 colours chosen: Number of ways of selecting 2 colours = ${}^3C_2$. For the two colours that have been selected, say A and B, there are two arrangements possible – ABAB or BABA. Total number of flags with 2 colors = ${}^3C_2 \times 2 = 3 \times 2 = 6$

Scenario II: All 3 colours being chosen. Any one colour will have to be repeated. Selecting the one colour that repeats can be done in ${}^3C_1$ ways. Post this, we have AABC. This can be rearranged in $\frac{4!}{2!}$ ways. Of these there will be 3! ways when the two A's appear together. So, the number of ways where 2 A's do not appear together will be 12 – 6 = 6. These are ABAC, ABCA, ACAB, ACBA, BACA, CABA. Total number of possible flags where three colours are chosen = ${}^3C_1 \times 6 = 18$ ways

So, there are 6 + 18 = 24 different flags possible totally.

This can also be done with another approach.

Let the four bands be called ABCD.

A can be selected in 3 ways. It could be blue, green or yellow.

B can be any of the three colours except A. So, there are 2 possible options for B.

C can be any of the three colours except B. So, there are 2 possible options for C.

D can be any of the three colours except C. So, there are 2 possible options for D.

Total number of options = 3 × 2 × 2 × 2 = 24 ways

Answer choice (c)

8. An equation $ax^2 + 8x + c = 0$ has two distinct real roots. If a, c are positive integers less than 11, how many values can (a, c) take?

CAT LEVEL 1

Solution

The equation has distinct real roots

$\Rightarrow b^2 - 4ac > 0$

$\Rightarrow b = 8.$

$\Rightarrow 64 - 4ac > 0$

$\Rightarrow 4ac < 64$

$\Rightarrow ac < 16$

If a = 1, c can be 1, 2, 3, 4, 5, 6, 7, 8, 9, or 10 – 10 possibilities

When a = 2, c can be 1, 2, 3, 4, 5, 6, or 7 → 7 possibilities

When a = 3, c can be 1, 2, 3, 4, 5 → 5 possibilities

When a = 4, c can be 1, 2, 3 → 3 possibilities

When a = 5, c can be 1, 2, 3 → 3 possibilities

When a = 6, c can be 1, 2 → 2 possibilities

When a = 7, c can be 1, 2 → 2 possibilities

When a = 8, c can be 1 → 1 possibility

When a = 9, c can be 1 → 1 possibility

When a = 10, c can be 1 → 1 possibility

Totally 35 pairs of values

Answer choice (d)

9. 2a + 5b = 103. How many pairs of positive integer values can a, b take such that a > b?

CAT LEVEL 1

SOLUTION

Let us find the one pair of values for a, b. a = 4, b = 19 satisfies this equation. $2\times4+5\times19=103$. Now, if we increase ‘a’ by 5 and decrease ‘b’ by 2 we should get the next set of numbers. We can keep repeating this to get all values.

Let us think about why we increase ‘a’ by 5 and decrease b by 2. a = 4, b = 19 works. Let us say, we increase ‘a’ by n, then the increase would be 2n. This has to be offset by a corresponding decrease in b. Let us say we decrease b by ‘m’. This would result in a net drop of 5m. In order for the total to be same, 2n should be equal to 5m. The smallest value of m, n for this to work would be 2, 5.

a = 4, b = 19

a = 9, b = 17

a = 14, b = 15

..

And so on till

a = 49, b = 1

We are also told that ‘a’ should be greater than ‘b’, then we have all combinations from (19, 13) … (49, 1). 7 pairs totally.

Answer choice (a)

10. Using the vertices of a regular hexagon as vertices, triangles of how many different areas can be formed?

CAT LEVEL 2

SOLUTION

Let us number the vertices from 1 to 6. There are three different types of triangles that can be formed.

Type I: (1, 2, 3): This type of triangle is also seen with (2, 3, 4), (3, 4, 5) etc.

Type II: (1, 2, 4): (1, 2, 5) has the same shape and area. So, do (2, 3, 5), (3, 4, 6) etc.

Type III: (1, 3, 5): (2, 4, 6) has the same shape and area.

So, there will be 3 triangles of different areas that can be formed from a regular hexagon.

There are 6 triangles of Type I, 12 of type II and 2 of type III, adding up to 20 triangles totally.

The total number of triangles possible = ${}^6C_3 = 20$.

Answer choice (a)

11. Diagonals of a square ABCD of side 35 cms intersect at O. A rhombus of perimeter 52 exists such that P lies on AO, Q on BO, R on CO and S on DO. What is the maximum number of circles with integer radii that can be drawn with center O such that the rhombus is inside the circle and the circle is inside the square?

CAT LEVEL 2

SOLUTION

We are trying to draw circles that are inside the square. So, radius should be less than

$\frac{35}{2} = 17.5$. Or, maximum integer radius should be 17.

We are trying to draw circles such that the rhombus should be inside the circle. So, the diameter of the circle should be greater than the longest diagonal of the rhombus. Or, the longer diagonal of the rhombus should be as short as possible. For a rhombus of the given perimeter, this is possible only if it is a square. A square of perimeter 52cm will have diagonals of length $13\sqrt{2}$.

Diameter of circle > $13\sqrt{2}$

Or radius > $\frac{13}{\sqrt{2}}$

Or, radius ≥ 10

Possible values of radius = 10, 11, 12, 13, 14, 15, 16, 17 = 8 different values.

Answer choice (b)

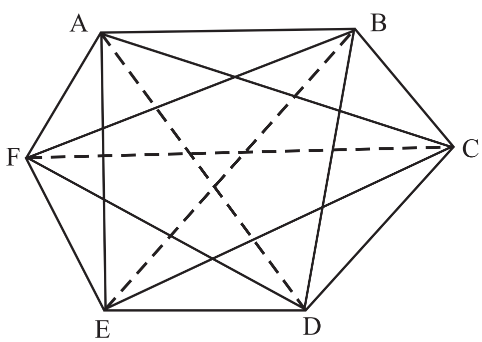
12. The diagonals of Hexagon intersect at n distimct points inside the hexagon. What is the maximum value n can take?

CAT LEVEL 2

SOLUTION

In any hexagon, there are $n\left(\frac{n-3}{2}\right) = 9$ diagonals.

First let us draw the diagonals and try to visualise this diagram



There are six 'short' diagonals AC, AE, CE, BD, BF, and DF. These intersect with other diagonals at 3 points each.

There are 3 long diagonals – AD, BE and CF. These intersect with other diagonals at 4 points each.

Note that the 'short' diagonals need not be shorter than the 'long' diagonal.

So, the total number of points of intersection should be 6 × 3 + 3 × 4 = 30. But in this case, we would count every point of intersection twice. So, number of points of intersection would be exactly half of this $= \frac{30}{2} = 15$ points.

Answer choice (c)

13. Consider a circle of radius 6 cms. What is the maximum number of chords of length 6 cms that can be drawn in the circle such that no two chords intersect or have points of contact?

CAT LEVEL 1

SOLUTION

First let us think about the angle subtended at the center by this chord. In a circle of radius 6cms, a chord of length 6 cms subtends an angle of 60° at the center. This chord along with the two radii forms an equilateral triangle.

So, we can place six such equilateral triangles at the center to account for 360°. This would form a regular hexagon. But in this scenario, the chords would have points of contact. So, the maximum number of chords that can be drawn such that there are no points of contact is 5.

More generally, if a chord makes an angle q at the center, then we can draw $\frac{360}{q}$ such chords around the circle. The maximum number of chords that can be drawn such that they do not touch each other = $\frac{360}{q} - 1$.

If $\frac{360}{q}$ is not an integer, then the maximum number of chords that can be drawn such that they do not touch each other = $\left[\frac{360}{q}\right]$, where [x] is the greatest integer less than or equal to x.

Answer choice (b)

14. x (x – 3) (x +2) < 200, and x is an integer such that |x| < 20, how many different values can n take?

CAT LEVEL 2

SOLUTION

Let us start with a trial and error. The expression is zero for x = 0, x = 3 and x = –2

x = 3, the above value = 0

x = 4, the above value would be 4 × 1 × 6 = 24

x = 5, the above value would be 5 × 2 × 7 = 70

x = 6, the above value would be 6 × 3 × 8 = 144

$x = 7$, the above value would be $7 \times 4 \times 9 > 200$

So, the equation holds good for $x = 3, 4, 5, 6$.

For $x = 2$ and 1, the above value is negative.

So, the above inequality holds good for $x = 6, 5, 4, 3, 2, 1, 0$.

For, $x = -1$, the value would be $-1 \times -4 \times 1 = 4$.

For, $x = -2$, the value would be 0.

So, this works for $x = 6, 5, 4, 3, 2, 1, 0, -1, -2$.

For, $x = -3$, the expression is negative, so holds good. For all negative values ≤ -3, this holds good. The smallest value x can take is –19.

So, the above inequality it holds good for –19, –18, –17…..–1, 0, 1 ……6, a total of 26 values.

Answer choice (c)

15. In how many ways can we pick three cards from a card pack such that they form a sequence of consecutive cards, not all cards belong to the same suit, and nor do all cards belong to distinct suits? Consider Ace to be the card following King in each suit. So, Ace can be taken to precede '2' and succeed 'King'. So, QKA would be a sequence, so would be A23. However, KA2 is not a sequence.

CAT LEVEL 2

SOLUTION

First let us see how many sequences of 3 we can form. We can have A23, 234…..JQK, QKA – a total of 12 sets of 3.

If cards should not be of the same suit, and nor should all three be of different suits, then we should have two cards from one suit and one from another.

So, cards should be from two suits. The two suits can be selected in $^4C_2$ ways. Now, from these two suits, one suit should have two cards. The suit that has two cards can be selected in $^2C_1$ ways. Now, out of the three cards, the two cards that have to be from the suit that repeats can be selected in $^3C_2$ ways.

So, total number of possibilities = $12 \times {^4C_2} \times {^2C_1} \times {^3C_2} = 12 \times 6 \times 2 \times 3 = 432$.

Answer choice (c)

16. Given that $|k| < 15$, how many integer values can k take if the equation $x^2 - 6|x| + k = 0$ has exactly 2 real roots?

CAT LEVEL 2

SOLUTION

The equation can be rewritten as $|x|^2 - 6|x| + k = 0$. This is a quadratic in $|x|$. This can have 2 real roots, 1 real root or 0 real roots.

If we have $|x|$ = positive value, we have two possible values for x.

If we have $|x|$ = negative value, we have no possible values for x.

If we have $|x| = 0$, we would have 1 possible value for x.

So, for the equation to have 2 values of x, we should have 1 positive root for $|x|$.

Scenario I: $|x|^2 - 6|x| + k$ has exactly one real root (and that root is positive). $b^2 - 4ac = 0 \Rightarrow k = 9$. If $k = 9$, $|x| = 3$, x can be 3 or –3

Scenario II: $|x|^2 - 6|x| + k$ has two real roots and exactly one of them is positive. This tells us that the product of the roots is negative. => k has to be negative. $|K|$ has to be less than 15.

$\Rightarrow$ k can take values –14, –13, –12, ….–1 14 different values

Total possibilities = –14, –13, –12, ….–1 and $k = 9$; 15 different values

Answer choice (a)

17. In how many ways can 6 boys be accommodated in 4 rooms such that no room is empty and all boys are accommodated?

CAT LEVEL 2

SOLUTION

No room is empty, so the boys can be seated as 1113 in some order or 1122 in some order.

Scenario I: 1113. We can do this as a two–step process.

Step I: Select the three boys – $^6C_3$.

Step II: Put the 4 groups in 4 rooms – 4! ways

Total number of ways = 20 × 4!

Scenario I: 1122. This is slightly tricky. So let us approach this slightly differently.

Step I: Let us select the 4 people who are going to be broken as 2 + 2; this can be done in $^6C_4$ ways. Now, these $^6C_4$ groups of 4 can be broken into 2 groups of two each in $\frac{^4C_2}{2}$ ways. So, the total number of ways of getting 2 groups of 2 is $^6C_4 \times \frac{^4C_2}{2} = 15 \times \frac{6}{2} = 45$ ways

Step II: Now, we need to place 2, 2, 1, 1 in four different groups. This can be done in 4! ways.

The total number of ways = 20 × 4! + 45 × 4! = 4! (20 + 45) = 24 × 65 = 1560 ways.

Answer choice (c)

18. In how many ways can 4 boys and 4 girls be made to sit around a circular table if no two boys sit adjacent to each other?

CAT LEVEL 2

SOLUTION

No two boys sit next to each other => Boys and girls must alternate. As they are seated around a circular table, there is no other possibility.

Now, 4 boys and 4 girls need to be seated around a circular table such that they alternate. Again, let us do this in two steps.

Step I: Let 4 boys occupy seats around a circle. This can be done in 3! ways.

Step II: Let 4 girls take the 4 seats between the boys. This can be done in 4! ways.

Note that when the girls go to occupy seats around the table, the idea of the circular arrangement is gone. Girls occupy seats between the boys. The seats are defined as seat between B1 & B2, B2 & B3, B3 & B4 or B4 & B1. So there are 4! ways of doing this.

Total number of ways = 3! × 4! = 6 × 24 = 144

Answer choice (c)

19. John extracts three letters from the word 'ACCEDE' and makes words out of them, how many different words can he generate?

CAT LEVEL 2

SOLUTION

John can extract three distinct letters or, 2 of one kind and one different.

Scenario I: Three distinct letters

Step I: Some 3 of the 4 letters A, C, D, E can be selected. $^4C_3$

Step II: This can be rearranged in 3! ways.

Total number of ways = $^4C_3$ × 3! = 4 × 6 = 24.

Scenario II:

Step I

(i) 2 C's and one of A, D or E or

(ii) 2 E's and one of A, C or D. 6 possibilities totally

Step II: This can be rearranged in $\frac{3!}{2!}$ ways

Total number of ways = 6 × 3 = 18

24 + 18 = 42

Answer choice (d)

20. Joseph extracts three letters from the word 'RENEGED' and makes words out of these 3 letters. How many such words can he generate?

CAT LEVEL 2

SOLUTION

Joseph can extract three distinct letters or, 2 of one kind and one different, or all 3 being the same letter.

Scenario I: Three distinct letters

Step I: Any 3 of the 5 letters R, E, N, G, D. $^5C_3$

Step II: This can be rearranged in 3! = 6 ways.

Total number of ways = $^5C_3$ × 3! = 10 × 6 = 60

Scenario II:

Step I: 2 E's and one of R, G, D or N. 4 ways of selecting one of the other 4 letters.

Step II: This can be rearranged in $\frac{3!}{2!}$ = 3 ways

Total number of ways = 4 × 3 = 12

Scenario III: All three being the same letter. All three can be 'E'. There is only one word.

60 + 12 + 1 = 73

Answer choice (d)

21. If we listed all numbers from 100 to 10,000, how many times would the digit 3 be printed?

CAT LEVEL 2

SOLUTION

We need to consider all three digit and all 4-digit numbers.

Three–digit numbers: A B C. 3 can be printed in the 100's place or 10's place or unit's place.

- **100's place:** 3 B C. B can take values 0 to 9, C can take values 0 to 9. So, 3 gets printed in the 100's place 100 times.
- **10's place:** A 3 C. A can take values 1 to 9, C can take values 0 to 9. So, 3 gets printed in the 10's place 90 times.
- **Unit's place:** A B 3. A can take values 1 to 9, B can take values 0 to 9. So, 3 gets printed in the unit's place 90 times.

So, 3 gets printed 280 times in 3–digit numbers.

Four–digit numbers: A B C D. 3 can be printed in the 1000's place, 100's place or 10's place or unit's place.

- **1000's place:** 3 B C D. B can take values 0 to 9, C can take values 0 to 9, D can take values 0 to 9. So, 3 gets printed in the 100's place 1000 times.
- **100's place:** A 3 C D. A can take values 1 to 9, C & D can take values 0 to 9. So, 3 gets printed in the 100's place 900 times.
- **10's place:** A B 3 D. A can take values 1 to 9, B & D can take values 0 to 9. So, 3 gets printed in the 10's place 900 times.
- **Unit's place:** A B C 3. A can take values 1 to 9, B & C can take values 0 to 9. So, 3 gets printed in the unit's place 900 times.

3 gets printed 3700 times in 4–digit numbers.

So, there are totally 3700 + 280 = 3980 numbers.

The alternative, much simpler way, would be to count the number of ways 3 would be printed while printing numbers from 1 to 10000, and then subtract the number of ways 3 would get printed from 1 to 100 from this number.

Number of ways 3 would be printed while printing numbers from 1 to 10000: 4000. 3 would get printed 1000 times in the units place, 1000 times in the tens place, 1000 times in the hundreds place and 1000 times in the thousands place.

Number of ways 3 would be printed while printing numbers from 1 to 100: 20. 10 times each in the units and tens place.

Answer = 4000 - 20 = 3980. As we have mentioned before, sometimes solving by a circuitous route could be instructive. So, we will continue to take detours like these.

Answer choice (a)

22. How many odd numbers with distinct digits can be created using the digits 1, 2, 3, 4, 5 and 6?

CAT LEVEL 2

SOLUTION

- Single digit numbers: 1, 3 or 5: Three numbers
- Two–digit numbers: Units digit = 1, 3 or 5. For the tens' digit, there are 5 choices {anything apart from what went into the units digit}. So, there will be 3 × 5 = 15 such numbers.
- Three–digit numbers: Units digit = 1, 3 or 5. For the tens' digit, there are 5 choices {anything apart from what went into the units digit}. For the 100s' digit, there are 4 choices {anything apart from what went into the units digit or tens digit}. So, there will be 3 × 5 × 4 = 60 such numbers.
- 4–digit numbers: There will be 3 × 5 × 4 × 3 = 180 numbers.

- 5–digit numbers: There will be $3 \times 5 \times 4 \times 3 \times 2 = 360$ numbers.
- 6–digit numbers: There will be $3 \times 5 \times 4 \times 3 \times 2 \times 1 = 360$ numbers.
- Total number of numbers = $360 + 360 + 180 + 60 + 15 + 3 = 978$.

Answer choice (c)

23. How many 5–digit numbers with distinct digits can be created with digits 1, 2, 3, 4, 5, 6 such that the number is multiple of 12?

CAT LEVEL 2

SOLUTION

For a number to be a multiple of 12, it has to be a multiple of 3 and 4.

For the number to be a multiple of 3, the sum of the digits should be a multiple of 3. The sum of all 6 digits = 21. This is a multiple of 3. So, if the sum of 5 digits has to be a multiple of 3, we need to drop one multiple of 3 from this.

So, the 5 distinct digits that can go into forming the number can be 1, 2, 3, 4, 5 or 1, 2, 4, 5, 6.

Numbers with digits 1, 2, 3, 4 and 5: For the number to be a multiple of 4, the last two digits should be a multiple of 4. The last two digits can be 12, 32, 52 or 24.

When the last two digits are 12: The number is __ __ __ 12. The first 3 digits have to be 3, 4, 5 in some order. There are 3! such numbers possible. Or, there are 6 numbers in this list.

So, total number of numbers possible with the digits 1, 2, 3, 4 and 5 = $6 \times 4 = 24$

Numbers with digits 1, 2, 4, 5 and 6: For the number to be a multiple of 4, the last two digits should be multiple's of 4. The last two digits can be 12, 32, 24, 64, 16 or 56.

With each of these as the last two digits, we can have 3! or 6 numbers.

So, the total number of numbers possible with the digits 1, 2, 4, 5 and 6 = $6 \times 6 = 36$.

The Total number of numbers = $24 + 36 = 60$.

Answer choice (b)

24. How many 6 letter words with distinct letters exist that have more vowels than consonants?

CAT LEVEL 1

SOLUTION

- 4 vowels and 2 consonants: ${}^5C_4 \times {}^{21}C_2 \times 6!$
- 5 vowels and 1 consonants: ${}^5C_5 \times {}^{21}C_1 \times 6!$

Total number of words = ${}^5C_4 \times {}^{21}C_2 \times 6! + {}^5C_5 \times {}^{21}C_1 \times 6! = 6!\,(5 \times 210 + 1 \times 21) = 6! \times 1071$

Answer choice (a)

25. How many 4 letters words can be created with more consonants than vowels?

CAT LEVEL 1

SOLUTION

- 4 consonants and 0 vowels: 21^4
- 3 consonants and 1 vowel: This can happen in three different ways
- 3 distinct consonants and 1 vowel: ${}^{21}C_3 \times {}^5C_1 \times 4!$
- 2 consonants of which 1 occurs twice and 1 vowel: ${}^{21}C_2 \times {}^2C_1 \times {}^5C_1 \times \frac{4!}{2!}$
- 1 consonant that appears thrice and 1 vowel: ${}^{21}C_1 \times {}^5C_1 \times \frac{4!}{3!}$

Total number of words = $21^4 + {}^{21}C_3 \times {}^5C_1 \times 4! + {}^{21}C_2 \times {}^2C_1 \times {}^5C_1 \times \frac{4!}{2!} + {}^{21}C_1 \times {}^5C_1 \times \frac{4!}{3!}$

Alternatively, the above answer can be given as $21^4 + 21^3 * 20$. Try to figure out the logic behind that answer.

Answer choice (b)

26. All the rearrangements of the word "DEMAND" are written without including any word that has 2 D's appearing together. If these are arranged alphabetically, what would be the rank of "DEMAND"?

CAT LEVEL 2

SOLUTION

Number of rearrangements of word DEMAND = $\frac{6!}{2!} = 360$

Number of rearrangements of word DEMAND where 2 D's appear together = 5! = 120

Number of rearrangements of word DEMAND where 2D's do not appear together = 360–120 = 240

1. Words starting with 'A'; without two D's adjacent to each other
   - Words starting with A: $\frac{5!}{2!} = 60$
   - Words starting with A where 2 D's are together = 4! = 24
   - Words starting with 'A', without two D's adjacent to each other = 36
2. Next we have words starting with D.
   - Within this, we have words starting with DA: 4! words = 24 words
   - Then words starting with DE
     - Within this, words starting with DEA – 3! = 6 words
     - Then starting with DED – 3! = 6 words
     - Then starting with DEM
       - First word is DEMADN
       - Second is DEMAND

Rank of DEMAND = 36 + 24 + 6 + 6 + 2 = 74

Answer choice (b)

27. If letters of the word SLEEPLESS were arranged alphabetically, what rank would 'SLEEPLESS' hold?

CAT LEVEL 2

SOLUTION

Total number of words = $\frac{9!}{3!3!2!} = 5040$

- Words starting with E = $\frac{8!}{3!3!2!} = 1680$
- Words starting with L = $\frac{8!}{3!3!} = 1120$
- Words starting with P = $\frac{8!}{3!3!2!} = 560$
- Words starting with S = $\frac{8!}{3!3!2!} = 1680$

Words gone by thus far

Starting with E	1680
Starting with L	1120
Stating with P	560

Now, within the words starting with S

Words starting with SE = $\frac{7!}{2!2!2!} = 630$

Words starting with SL = $\frac{7!}{3!2!} = 630$, SLEEPLESS is within this list, so we need to drill further.

Words starting with SLEEE = $\frac{4!}{2!} = 12$ words

Words starting with SLEEL = $\frac{4!}{2!} = 12$ words

Words starting with SLEEPE = $\frac{3!}{2!} = 3$ words

Words gone thus far

Starting with E	1680
Starting with L	1120
Starting with P	560
Starting with SE	630
Starting with SLEEE	12
Starting with SLEEL	12
Starting with SLEEPE	3

Then comes SLEEPLESS (at long last): Rank = 4018

Answer choice (b)

28. If all words with 2 distinct vowels and 3 distinct consonants were listed alphabetically, what would be the rank of "ACDEF'?

CAT LEVEL 2

SOLUTION

The first word would be ABCDE. With 2 distinct vowels, 3 distinct consonants, this is the first word we can come up with.

Starting with AB, we can have a number of words.

AB __ __ __. The next three slots should have 2 consonants and one vowel. This can be selected in $^{20}C_2$ and $^4C_1$ ways. Then the three distinct letters can be rearranged in 3! ways.

Or, the number of words starting with AB = $^{20}C_2 \times {}^4C_1 \times 3! = 190 \times 4 \times 6 = 4560$

Next, we move on to words starting with ACB.

ACB __ __. The last two slots have to be filled with one vowel and one consonant. = $^{19}C_1 \times {}^4C_1$. This can be rearranged in 2! ways.

Or, the number of words starting with ACB = $^{19}C_1 \times {}^4C_1 \times 2 = 19 \times 4 \times 2 = 152$.

Next we move on words starting with ACDB. There are 4 different words on this list – ACDBE, ACDBI, ACDBO, ACDBU.

So far, number of words gone = 4560 + 152 + 4 = 4716

Starting with AB	4560
Starting with ACB	152
Starting with ACDB	4
Total words gone	4716

After this when we move to words starting with ACDE, the first possible word is ACDEB. After this we have ACDEF.

So, rank of ACDEF = 4718

Answer choice (c)

29. When a die is thrown 4 times in how many ways can we have at least one digit appearing exactly twice?

CAT LEVEL 2

SOLUTION

Two scenarios are possible.

Scenario I: One digit appears twice accompanied with two distinct digits, AABC. $^6C_1 \times {}^5C_2 \times {}^4C_2 \times 2!$.

- Selecting the digit that appears twice
- Selecting the other two digits
- Selecting the two throws where the digit appearing twice appears
- Number of arrangements for the final two throws

Scenario II: Two digits appear twice each, AABB.

$$\frac{^6C_2 \times 4!}{2!2!}.$$

- Selecting the digits that appear twice
- Number of arrangements for AABB

Total number of ways = $^6C_1 \times {}^5C_2 \times {}^4C_2 \times 2! + {}^6C_2 \times \frac{4!}{2!2!}$

Answer choice (c)

30. When a die is rolled 3 times, how many possibilities exist such that each throw results in a number that is at least as high as that of the preceding throw?

CAT LEVEL 2

SOLUTION

If the first throw were 6, there is only one possibility = 666

If the first throw were 5, there are three possibilities for the next two throws = 55, 56 and 66

If the first throw were 4, the next two throws can be 44, 45, 46, 55, 56, and 66 = 6 possibilities

If the first throw were 3, the next two throws can be 33, 34, 35, 36, 44, 45, 46, 55, 56, 66 = 10 possibilities; this is nothing but 4 + 3 + 2 + 1.

> GURU GYAN
>
> It is important to pick this pattern. The {1, 3, 6, 10, 15, 21…} pattern is very common in counting. In this sequence each term t_n is nothing but the sum of natural numbers till n.

If the first throw were 2, the next two throws can be obtained in 5 + 4 + 3 + 2 + 1 = 15 ways.

If the first throw were 1, the next two throws can be obtained in 6 + 5 + 4 + 3 + 2 + 1 = 21 ways.

Total number of possibilities = 21 + 15 + 10 + 6 + 3 + 1 = 56 ways.

Answer choice (a)

31. When a die is rolled thrice, in how many outcomes will we have the product of the throws and sum of the throws to be even numbers?

CAT LEVEL 2

SOLUTION

The product has to be even => There should be at least one even number.

The sum has to be even => There are either 3 even numbers or 2 odd numbers, and 1 even number.

So, both conditions put together, there are two possibilities – all three throws being even, there being 2 odd throws and 1 even throw.

- **Scenario I:** All three even: 3 × 3 × 3 = 27 possibilities
- **Scenario II:** Two odd and one even, the two odd slots can be selected in $^3C_2$ = 3 ways. After these have been selected, the number of options for each throw is 3. Number of ways = 3 × 3 × 3 ×3 = 81 ways

So, overall number of ways = 27 + 81 = 108 ways.

Answerchoice (a)

32. A string of n consecutive even natural numbers has 23 multiples of 14 in it. How many of the following could be true?

1. n = 157
2. Of these n numbers, exactly 80 are multiples of 4.
3. Of these n numbers, more than 50 are multiples of 3.

CAT LEVEL 2

SOLUTION

Let us start with a simple example. Let us say we list down even numbers starting from 2.

So, our sequence reads {2, 4, 6, 8, 10,……..2n}.

Now, this list should have 23 multiples of 14. So, the smallest value 2n could take for this to be true is 14 × 23, which is 322.

So, if our sequence ran from {2, 4, 6, 8, ….322} it would have 23 multiples of 14 in it – all multiples from 14 to 322. In this case, the value of n would be 161.

However, we can see that a sequence running from {2, 4, 6,…334} would also have 23 multiples of 14 in it, and so would a sequence that runs as {14, 16, 18,….322}.

We can arguably have a sequence that starts and ends with a multiple of 14 – this type of sequence would have 155 elements (161 – 6). Or, have a sequence that starts with a number that is of the type 14n + 2 and ends with a number of the type 14K –2 (so that we include maximum number of non–multiples of 14 in the list); this type of sequence will have 167 elements (161 + 6). So, n can range from 155 to 167.

Now, to the statements

1. n = 157, n can range from 155 to 167, so this could be true.
2. Of these n numbers, exactly 80 are multiples of 4. If there are 160 consecutive even numbers, exactly 80 would be divisible by 4. So, this could be true.
3. Of these n numbers, more than 50 are multiples of 3. If we take 150 consecutive even numbers, 50 will be multiples of 3. N is definitely more than 150. So, there will definitely be more than 50 multiples of 3.

So, all three statements are true.

Answer choice (d)

33. The product of the digits of a 5–digit number is 1800. How many such numbers are possible?

CAT LEVEL 2

SOLUTION

1800 = $2^3 \times 3^2 \times 5^2$. From this it is clear that two digits have to be 5. The remaining three digits multiply to give 72.

72 can be written as a product of 3 digits in the following ways.

- 1 × 8 × 9
- 2 × 4 × 9
- 2 × 6 × 6
- 3 × 3 × 8
- 3 × 4 × 6

So, the digits can be

1 8 9 5 5: Number of numbers = $\frac{5!}{2!}$ = 60

2 4 9 5 5: Number of numbers = $\frac{5!}{2!}$ = 60

2 6 6 5 5: Number of numbers = $\frac{5!}{2!2!}$ = 30

3 3 8 5 5: Number of numbers = $\frac{5!}{2!2!}$ = 30

3 4 6 5 5: Number of numbers = $\frac{5!}{2!}$ = 60

Total number of numbers = 60 × 3 + 30 × 2 = 240

Answer choice (d)

34. The sum of three distinct positive integers is 16. How many values can a x b x c take?

CAT LEVEL 2

SOLUTION

a + b + c = 16

Let the smallest value be a.

If a = 1, b + c = 15. This can be done in 6 ways –2 + 13, 3 + 12, 4 + 11, 5 + 10, 6 + 9, 7 + 8

If a = 2, b + c = 14. This can be done in 4 ways – 3 + 11, 4 + 10, 5 + 9, 6 + 8

If a = 3, b + c = 13. This can be done in 3 ways – 4 + 9, 5 + 8, 6 + 7

If a = 4, b + c = 12. This can be done in 1 way – 5 + 7 a cannot be 5 or more as we have assumed a to be the smallest number.

Totally, there are 6 + 4 + 3 + 1 = 14 ways.

a × b × c is distinct for all of these ways. So, we do not have to worry about that. 14 different products can be formed.

Answer choice (c)

35. Product of 3 distinct positive numbers is 120. How many such sets are possible?

CAT LEVEL 2

SOLUTION

Again, let us take a × b × c = 120. a < b < c. In all of these questions, it helps to count with a pattern.

Let a = 1, b × c = 120. 120 = 2^3 × 3 × 5, which has 16 factors. Or, 120 can be written as a product of two natural numbers b × c in 8 ways. Of these 8 ways, one will be 1 × 120. Subtracting this one way, we have 7 pairs with a = 1.

a = 2; b × c = 60; b, c > 2

3 × 20

4 × 15

5 × 12

6 × 10

4 options

a = 3, b × c = 40 ; b, c > 3

4 × 10

5 × 8

a = 4, b × c = 30 ; b, c > 4

5 × 6

Totally there are 7 + 4 + 2 + 1 = 14 options.

Answer choice (b)

36. Set A has element {a, b, c, d, e} Set B = {1, 2, 3}. How many onto functions exist from set A to set B such that f(a) = 1.

CAT LEVEL 2

SOLUTION

First let us compute the number of onto functions from Set A to Set B.

Step I: Number of functions from Set A to Set B = 3^5 = 243. Each element in Set A has 3 options which it can be mapped to. So, the total number of sets = 3^5.

Step II: Let us calculate the number of ways in which a function can be not onto.

This can happen in two scenarios.

Scenario I: Elements in Set A are mapped to exactly 1 element in Set B. This can be done in 3 ways. All elements to either 1, 2 or 3.

Scenario II: Elements in Set A are mapped to exactly 2 of the elements in Set B, for instance {a, b, c, d, e} mapped to {1, 2}. Now, this can happen in 2^5 ways. But within this 2^5 ways, we would count the two ways where all elements are mapped to 1 and all elements are mapped to 2 as well (the instances that have already been accounted for in Scenario I). So, the number of ways in which a function can be mapped from {a, b, c, d, e} to {1, 2} such that both 1 and 2 have some element mapped to them = $2^5 – 2 = 30$.

The function could have been defined from Set A to {1, 2} or {2, 3} or {1, 3}. So, total number of possibilities = 30 × 3 = 90.

Total number of non–onto functions = 90 + 3 = 93.
Total number of onto functions = 243 – 93 = 150.

Exactly one–third of these will have f(a) = 1. Or, there are 50 onto functions possible such that f(a) = 1.

Answer choice (c)

37. Set A has elements {a, b, c, d, e, f}. How many subsets of A have element ‘a’, do not have element b and have an even numbers of elements?

CAT LEVEL 1

SOLUTION

A subset should have element a and it should have an even number of elements. So, it can have 2, 4 or 6 elements. But since it cannot have b, it can have only either 2 or 4 elements.

2–element subsets: There are 4 ways of doing this. Apart from a the subset can have c, d, e or f.

4–element subsets: Apart from ‘a’ the subset should have some 3 of c, d, e or f or ${}^4C_3$ subsets.

Totally 4 + 4 = 8 subsets are possible.

Answer choice (c)

38. How many rearrangements of the word EDUCATION are there where no two consonants are adjacent to each other?

CAT LEVEL 1

SOLUTION

Place the 5 vowels – AEIOU. This can be done in 5! ways.

Now, create one slot between every two vowels and one before all the vowels and one after all the vowels

__ E __ U __ A __ I __ O __

Of these 6 slots, some 4 have to be taken by consonants. This can be done in ${}^6C_4$ ways. Then the consonants can be arranged in 4! ways.

So, total number of outcomes = $5! \times {}^6C_4 \times 4!$

Answer choice (a)

39. Consider 24 points of which 7 lie on a straight line and 8 are on a different straight line. No other set of 3 points are collinear. How many triangles can be considered out of these 24 points?

CAT LEVEL 1

SOLUTION

Four scenarios are possible.

Scenario I: All 3 points of the triangle are from the 9 non–collinear points. ${}^9C_3 = 84$

Scenario II: 2 points are from the 9 non–collinear points and 1 from the remaining 15. ${}^9C_2 \times {}^{15}C_1 = 540$

Scenario III: 2 points from one of the two lines and 1 point from among the 9 non–collinear points. ${}^8C_2 \times {}^{16}C_1$ or ${}^7C_2 \times {}^{17}C_1 = 448 + 357 = 805$

Scenario IV: 1 point each from the two lines that contain collinear points and 1 point from among the 9 non–collinear points. ${}^8C_1 \times {}^7C_1 \times {}^9C_1 = 504$

Total scenarios = 84 + 540 + 805 + 504 = 1933

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The same answer can be obtained as $^{24}C_3 - {^7C_3} - {^8C_3}$, which is obviously the far more elegant method. But what is the joy in doing only by the elegant methods!

Answer choice (b)

40. How many numbers with distinct digits are possible, the product of whose digits is 28?

CAT LEVEL 2

SOLUTION

Two digit numbers: The two digits can be 4 and 7: Two possibilities 47 and 74

Three–digit numbers: The three digits can be 1, 4 and 7: 3! or 6 possibilities

We cannot have three digits as (2, 2, 7) as the digits have to be distinct.

We cannot have numbers with 4 digits or more without repeating the digits.

So, there are totally 8 numbers.

Answer choice (c)